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INFLUENCE CHART FOR MOMENTS IN RAILWAY RAILS

by
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conducted by
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ENGINEERING EXPERIMENT STATION
UNIVERSITY OF ILLINOIS
in cooperation with
THE UNIVERSITY OF ILLINOIS RESEARCH BOARD
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Abstract

This report contains a graphical method of analysis for determining the moments in a railway rail. The method, based on a beam on an elastic foundation analysis, requires use of an influence chart and requires the wheel loading configuration under consideration to be drawn to a scale of 1" equals 1' -0" and then be placed on the chart. Coefficients taken from the chart are then used to determine the moment at a given point in the rail. Changes in the track properties or finding the moment at another point in the rail are accomplished by changing the position of the scaled wheel loading configuration on the chart and noting new coefficients.

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1. INTRODUCTION AND PURPOSE

The current method of determining moments in a railroad rail was presented in the "Progress Report of the Special Committee to Report on Stresses in Railroad Track" of the American Railway Engineering Association*. The method is based on a beam on an elastic foundation analysis and is implemented by calculating a number of parameters. Then through use of a "Master Diagram" moment coefficients are found which are in turn related to the moments in a rail. The method is briefly described later in this paper.

The purpose of this paper is to set forth a graphical method for determining moments in a rail. The method requires that the wheel loading configuration be drawn to a scale of 1" equals 1'-0" and this scale drawing be placed on an "Influence Chart".

As will be seen, because of the graphical nature of this method, many calculations are automatically accomplished by changing the position of the scaled configuration on the chart whereas use of another method may necessitate many machine or hand computations to obtain the same results.

2. CONVENTIONAL METHOD-MASTER DIAGRAM

The theory for finding moments in a rail is based on the solution of the differential equation for a beam on an elastic foundation. For

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*Progress Report of the Special Committee to Report Stresses in Railroad Track - Volumn 19 Proceedings of the American Railway Engineering Association - pp. 875-1058.

a single wheel load, the solution for the moment, at a distance X from the load is:

$$M = P \sqrt[4]{\frac{EI}{64U}}\, e^{-X \sqrt[4]{\frac{U}{4EI}}} \left(\cos\left(X \sqrt[4]{\frac{U}{4EI}}\right) - \sin\left(X \sqrt[4]{\frac{U}{4EI}}\right)\right) \ldots\ldots (1)$$

For $X \geq 0$

WHERE: P = wheel load in pounds

E = the modulus of elasticity of the rail

I = the moment of inertia of the rail

U = the modulus of rail support

X = the distance from the load P

e = base of natural logarithms = 2.7183

To find the total moment due to more than one load, the moment contribution for each load could be superimposed. Obviously, equation (1) must be applied once for each load. This would be cumbersome for hand computation. However, equation (1) may be expressed graphically in a "Master Diagram" so that the moment equation becomes $M = Mo \times C$

WHERE: $Mo = .318PX_1$

$$X_1 = \frac{\pi}{4} \sqrt[4]{\frac{4EI}{U}}$$

C = a coefficient taken from the Master Diagram

π = a constant = 3.1416

Figure (1) shows the "Master Diagram". As an illustration of the use of this diagram, suppose one wished to calculate the moment at point A in Figure (2), due to the wheel loading configuration shown in the figure.

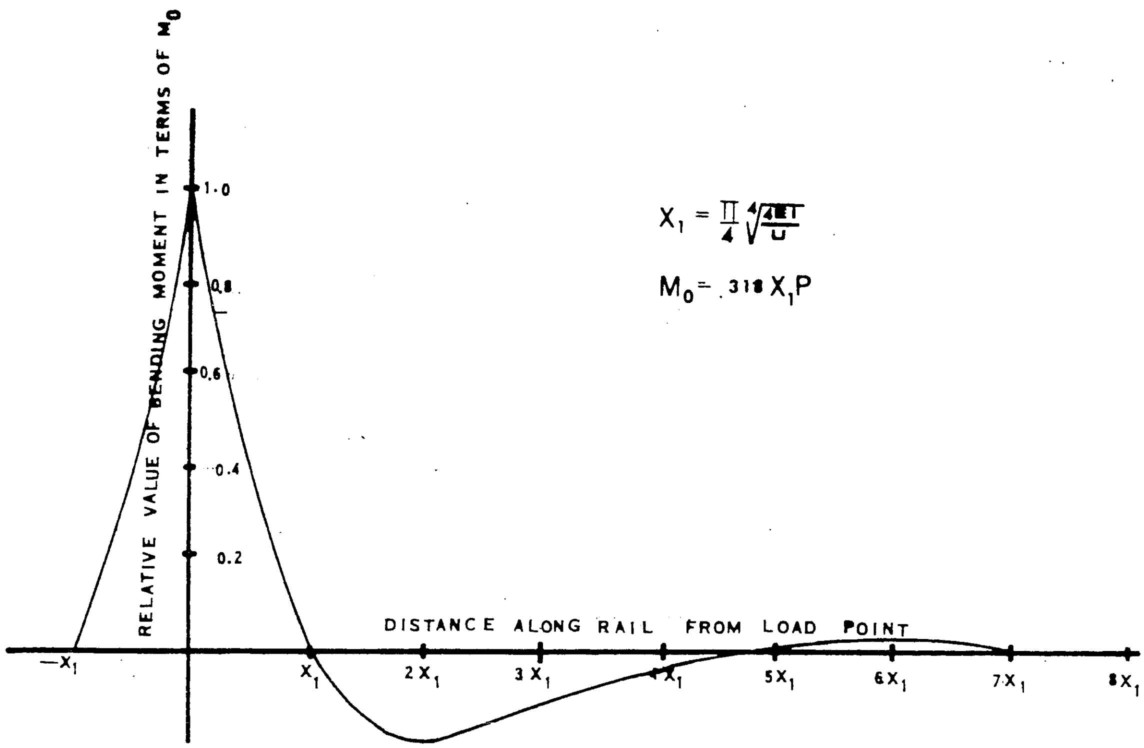

FIGURE 1 MASTER DIAGRAM MOMENT LINE

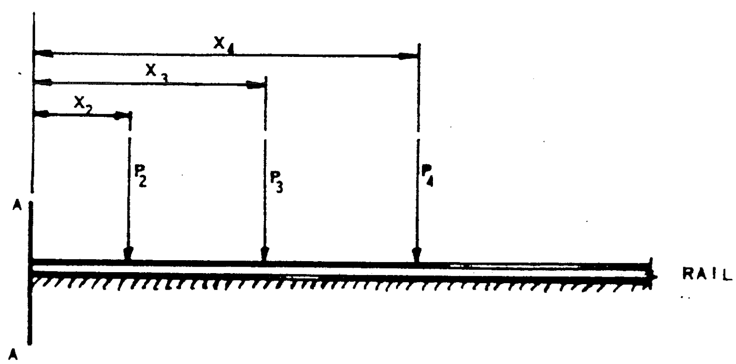

FIGURE 2 WHEEL LOADING CONFIGURATION

The procedure would be as follows:

(1) calculate $X_1 = \frac{\pi}{4}\sqrt{\frac{4EI}{U}}$

(2) calculate $Mo = 0.318\ P_1X_1$

(3) form the ratio X_2/X_1

(4) from the Master Diagram take the Relative Value of Bending Moment in Rail in terms of Mo (ordinate) for the abscissa found in (3).

(5) multiply the value in (4) by Mo

(6) repeat (2), (3), (4), (5), for X_3/X_1 and P_3

(7) repeat (2), (3), (4), (5), for X_4/X_1 and P_4

(8) step (5) in each case yields the moment contribution for the load under consideration. The sum of all moment contributions is the total moment. To find the moment at any other point, the above procedure is repeated using new values of X_2, X_3, and X_4.

3. GRAPHICAL METHOD-INFLUENCE CHART

If one glances back at equation (1), it will be observed that E, I, and U always appear as a ratio and never as separate entities. As E in most cases can be chosen equal to 30,000,000 psi, we can limit our discussion to the effect of I and U. (As will be seen later, the effect caused by an E other than 30,000,000 psi does not cause loss of generality in the method to be presented).

Further, for most track conditions the ratio of I/U will be between 0.01 and 0.12. Again one should note that I/U always appears as a ratio in equation (1), and therefore the moment in a rail due to any wheel configuration is dependent only on this ratio, and not on a specific value of I or U. Thus, if both I and U are increased or decreased proportionately, the moment in the rail will remain the same.

Finally, one should observe from equation (1), that the moment caused by a wheel load at a distance greater than 20' from the wheel load, becomes small.

Suppose now using E = 30,000,000 psi and I/U = 0.12 and P = 1 lb., the moment found for equation (1) is plotted as a function of the distance (in feet) from the load. The result is shown in Figure (3). (One should note that the ordinate becomes an influence coefficient and the moment due to a load of P lbs. is just P x the influence coefficient.)

Similarly, suppose for E = 30,000,000 psi and I/U = 0.01 and P = 1 lb., equation (1) is again plotted. The result will be as seen in Figure (4). Similar moment curves will occur for any other I/U between 0.01 and 0.12, E = 30,000,000 psi and P = 1 lb.

Now, instead of making many I/U graphs, suppose a three dimensional plot is made of the moment coefficient on the vertical axis, X, the distance from the load on one horizontal axis and I/U on the other horizontal axis. The result is shown in Figure (5).

Now suppose that a line is drawn through all points whose values are 15, 10, 5, 0, -1, -2, -3; this is also shown in Figure (5).

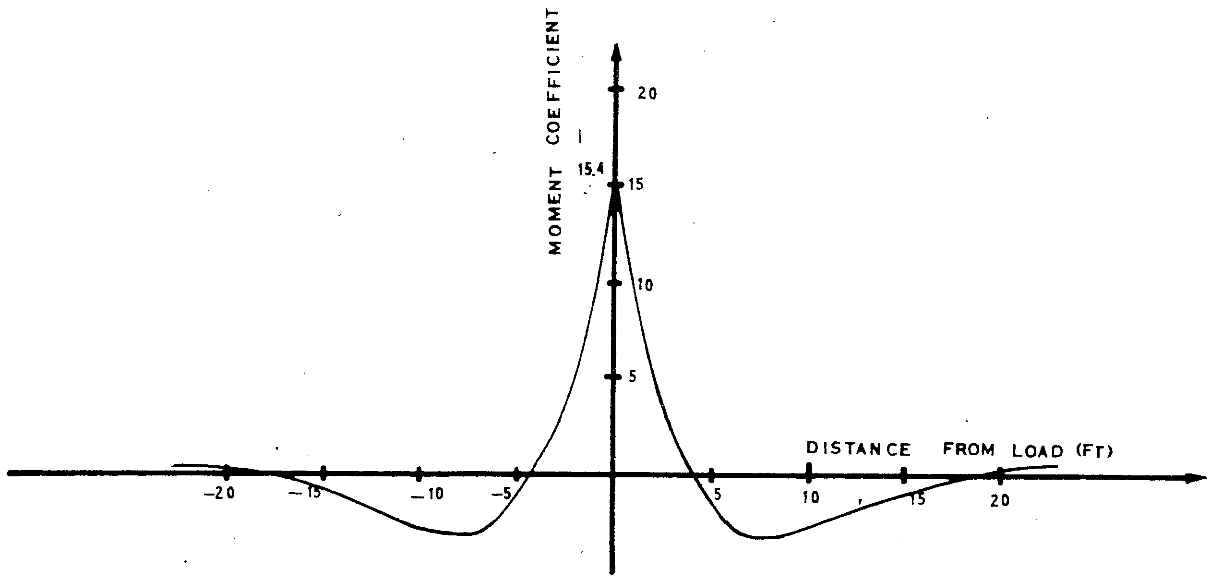

FIGURE 3 MOMENT INFLUENCE LINE FOR $\frac{I}{U} = 0.12$

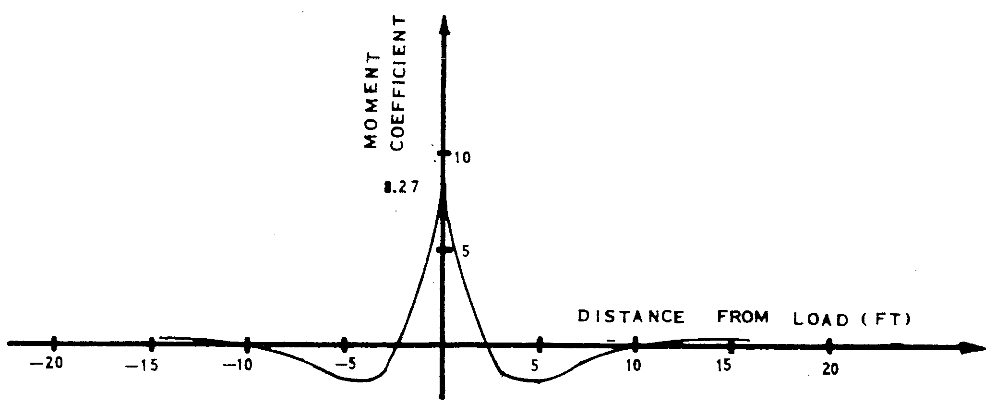

FIGURE 4 MOMENT INFLUENCE LINE FOR $\frac{I}{U} = 0.01$

Finally, if these lines are projected on a horizontal plane as shown in Figure (6), the projection becomes an "Influence Chart for Moments".

An influence chart visualized in the above manner accompanies this report. A description of the generation of the chart in mathematical terms may be found in the Appendix. The use of the chart requires that the I/U ratio is calculated and the wheel loading configuration scaled 1" equals 1'-0". The moment at some point X on the scaled wheel configuration diagram is found by laying the point X over the I/U value so that the diagram is perpendicular to the I/U axis at the I/U value under consideration. The moment coefficient value for each load is the value on the chart at the load in the scaled diagram. The total moment is the sum of load influence coefficient times the corresponding load described above.

One will note that once the wheel configuration has been scaled the computation will involve one division to calculate I/U and then one multiplication for each wheel load and one final summation to find the moment at any point. Finding the moments at other positions under the wheel loading configuration or at other value of I/U requires only a sliding of the scaled wheel configuration on the chart and repeating the above calculations.

4. AN EXAMPLE

For an illustration, suppose the moments under the wheel load P_1 and at "A" of the wheel configuration shown in Figure (7) are to be calculated using the influence chart for a 132 lb. rail ($I = 88.2$ in.^{4}),

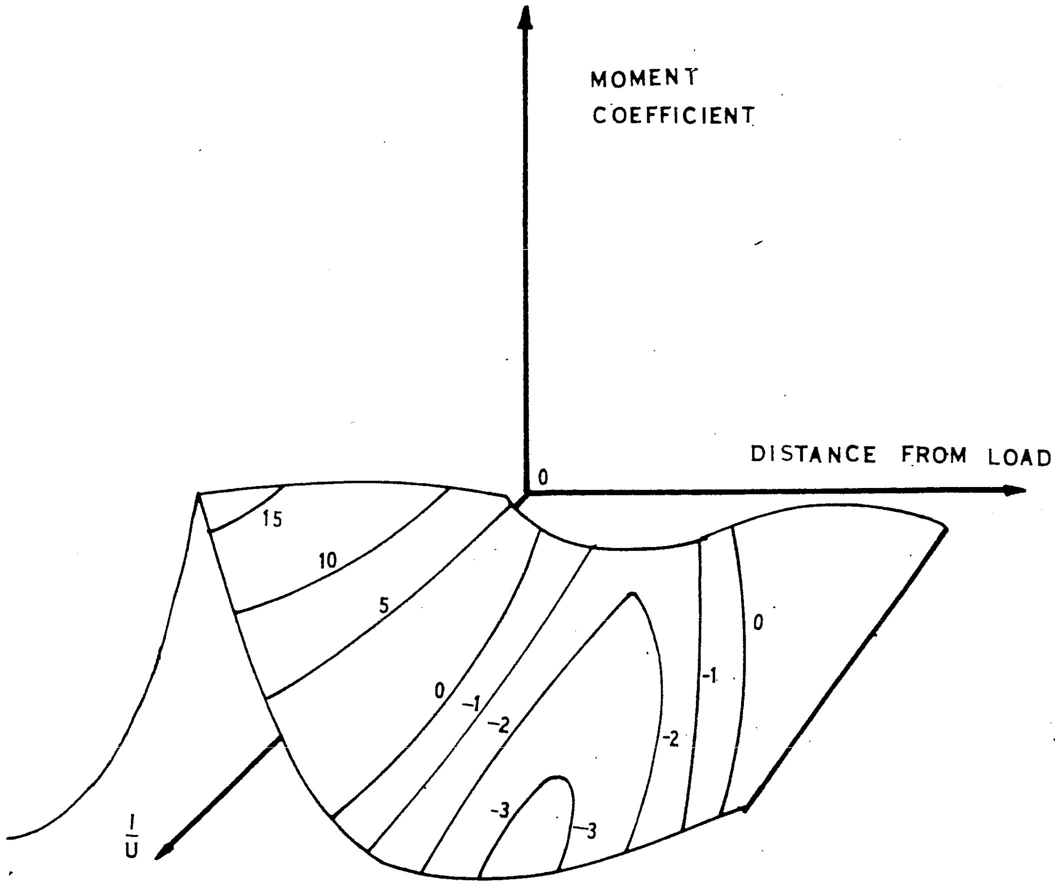

FIGURE 5 MOMENT INFLUENCE SURFACE

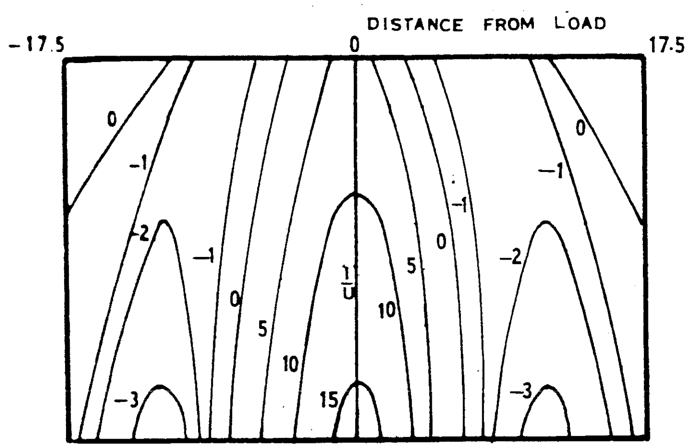

FIGURE 6 MOMENT INFLUENCE CHART

E of 30,000,000 psi and U equal to 2000#/in./in. The ratio I/U is then $\frac{88.2}{2000}$ = 0.0441. To find the moment under P_1 the wheel load would then be scaled 1" equals 1'-0" and then the scaled configuration would be placed on the influence chart as shown in Figure (8).

The moment coefficient taken from the chart for P_1 is 12.0 and for P_2 is -1.7. The total moment therefore is 12.0 x 31,500 + (-1.7) x 31,500 = 324,450 in-lbs. (The actual solution when run on a computer = 326,014 in-lbs.)

To find the moment under "A" the scaled wheel loading configuration is then placed as shown in Figure (9). The coefficients taken from the chart are for P_1 = 2.0 and P_2 = -2.0. The total moment is therefore (-2.0) x 31,500 + (-2.0) x 31,500 = -126,000 in lbs. (Actual solution when run on a computer = -123,954 in-lbs.)

5. VARIATION IN E

The influence chart was developed for an E = 30,000,000 psi. The question remains as to how the chart may be used if E was of a value other then 30,000,000 psi. A glance back at equation (1) shows that E always appears as a ratio with I and U i.e., EI/U or U/EI. Therefore to use the influence chart for a value of E other than 30,000,000 say E_1 the I/U ratio should be modified to $\frac{E_1}{30,000,000} \times \frac{I}{U}$. This new modified ratio would then be used as the I/U ratio on the influence chart. For instance, if in the previous example suppose E_1 was chosen to be 29,000,000 psi, then instead of referring to the I/U value of 0.0441 on the influence chart, the scaled wheel configuration should

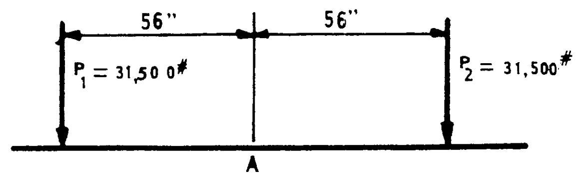

FIGURE 7 WHEEL LOADING CONFIGURATION

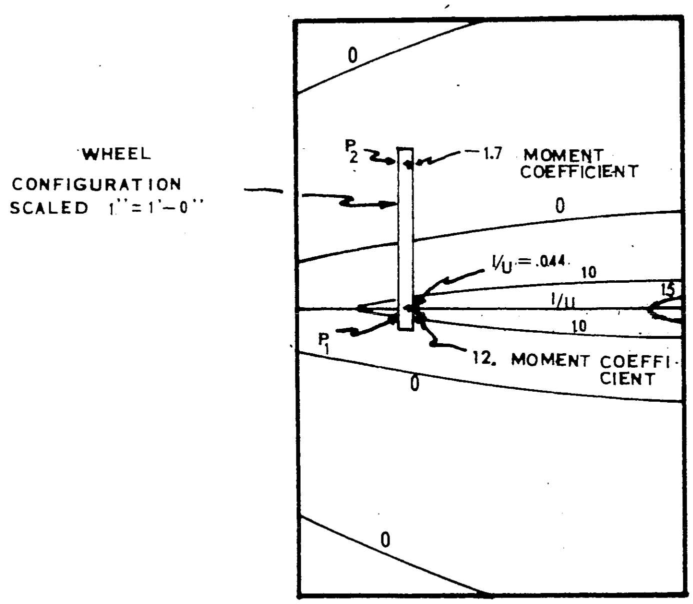

FIGURE 8 FINDING MOMENT COEFFICIENTS FOR MOMENT UNDER P_1

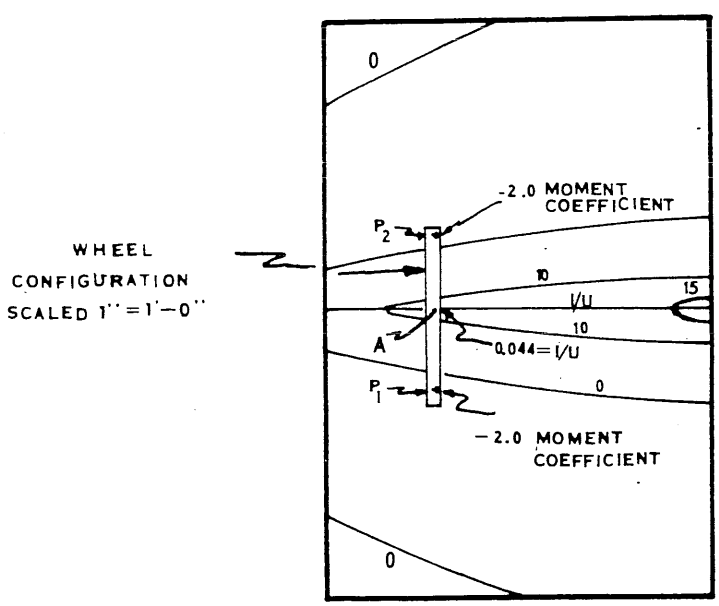

FIGURE 9 FINDING MOMENT COEFFICIENTS FOR MOMENT AT A

be placed at the I/U value of $\frac{29,000,000}{30,000,000} \times 0.0441 = 0.0426$.

6. ERRORS

The influence chart only shows values for the moment coefficient 17.5 feet from the Axis A-A. The question then arises as to the magnitude of the coefficients which lie off the influence chart. The maximum value of the influence coefficient off the chart is less then 0.13 for I/U less than 0.07. For I/U = 0.12 the maximum coefficient not shown is 0.32, however, beyond 20 feet from A-A the maximum coefficient is less then 0.14. In any case, these coefficients become relatively small and, except in very unusual circumstances, the effect of a wheel loading extending beyond the chart will be of minor consequence. As for other errors which arise because of the graphical procedure, one should assume that the method yields only two significant figures.

7. SUMMARY

This paper presents a graphical method of finding moments in a railroad rail arising out of the beam on an elastic foundation analysis. The method itself requires use of scaled wheel loading configuration, the accompanying influence chart and a few simple computations. An example is then presented as an illustration.

APPENDIX

GENERATION OF THE INFLUENCE CHART FOR MOMENTS IN RAILWAY RAILS

In an earlier section, it was stated that a three dimensional surface could be generated which described the variation of "X", "I/U" and "M" in equation (1) when "P" = 1 lb. and "E" = 30,000,000 psi. Lines could then be drawn through points of equal value on the surface and then these lines projected onto a horizontal plane. The projection became an influence chart for moments. The question now arises as to how the chart can be (and was) realized in more mathematical terms.

First one should note that the chart must be symmetric about the "I/U" axis. One can see this either physically or by means of equation (1). Thus the remaining discussion can focus on the development of one-half of the chart and the remainder of the chart can be developed by symmetry.

Secondly, if "P" is set equal to 1 lb. in equation (1), values then derived from the equation become influence coefficients. The moment due to a load of "P" lbs. is then just "P" times the influence coefficient. Since the chart is based on a "P" of 1 lb. and an "E" of 30,000,000 psi, the remaining discussion will assume the use of these values. In this light, "M" and the term "influence coefficient" are equal so they are used synonymously in the following discussion.

Bearing in mind the above preliminary remarks, the development of the influence chart is not difficult. Every line on the influence chart is a plot of "X" as the ordinate and the "I/U" ratio as the abscissa for "discrete" values of "M", using "P" = 1 lb. and "E" = 30,000,000 psi. The "discrete" values of "M" were chosen from 15.0 to -3.0, as these values were the range of the maximum and minimum values of the influence coefficients, for "I/U" between .01 and .12.

In the computer program, "M" and "I/U" were specified in equation (1) and then "X" was found by "Newton's Method". (Newton's Method" is a procedure for solving an equation. One "guesses" at a solution and then applies the procedure. The procedure is an iterative process which converges to the "true" solution. A description of "Newton's Method" may be found in many books on analytical geometry and calculus.) However, when setting equation (1) equal to a descrete value of "M" and solving for "X" a good guess for the starting value in "Newton's Method" should be made. This is because of the oscillatory nature of equation (1), and if a good initial starting value is not chosen, the method may converge to an erroneous value. (For example, there are theoretically an infinite number of "X's" for "M" = 0.0 but for the influence chart one is only interested in those "X" values less than 17.5 feet. One must therefore chose starting values so that "true" rather than erroneous values are found.) Finding good starting values is not a difficult task if one knows the shape of the function with which one is dealing. In the case under consideration, good starting values were found by making plots of "M" versus "X" for a given "I/U".

The computer program itself started by assuming an "I/U" = .12 (and "E" = 30,000,000 psi, "P" = 1 lb.) in equation (1). Then it read an "M" and a starting value for "X" in "Newton's Method". "Newton's Method" was then initiated by the program which caused "X" to converge to a "true" value. The value for "X" and "I/U" = .12 was stored and then "I/U" was decremented by .0001. The "X" found in the "I/U" = .12 case was then used as a starting value in "Newton"s Method" for the "I/U" = .12 - .0001 = .1199 case. Another "true" "X" was found and it was stored along with its corresponding "I/U" (i.e., "I/U" = .1199). The process whereby "I/U" is decremented, a starting value for "X" chosen to be the "X" value from the preceeding case, "Newton's Method" applied to find a "true" value and then storing the "X" and corresponding "I/U" value, was continued until "I/U" was decremented to .01, or if the "M" did not exist at the "I/U" under consideration. (For example, when "M" equals "5" the process stops when "I/U" reaches .01. On the other hand, however, when "M" equals "15" the procedure terminates at an "I/U" of about .11.)

Once the procedure had terminated, the stored arrays, i.e. the "X" and the "I/U" arrays were plotted using the "X" 's and the "I/U" 's as coordinates. The plot became a line on the influence chart. Its mirror image was then plotted as the chart is symmetrical. Finally, a new "M" and starting value for "X" was read, "I/U" set equal to .12 and the entire procedure restarted to yield another line. Lines were continually generated until there were no more "M's" to be read.